

General solutions for flat Friedmann universe filled by perfect fluid and scalar field with exponential potential

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Abstract. We study integrability by quadrature of a spatially flat Friedmann model containing both a minimally coupled scalar field φ with an exponential potential $V(\varphi) \sim \exp[-\sqrt{6}\sigma\kappa\varphi]$, $\kappa = \sqrt{8\pi G_N}$, of arbitrary sign and a perfect fluid with barotropic equation of state $p = (1 - h)\rho$. From the mathematical view point the model is pseudo-Euclidean Toda-like system with 2 degrees of freedom. We apply the methods developed in our previous papers, based on the Minkowsky-like geometry for 2 characteristic vectors depending on the parameters σ and h . In general case the problem is reduced to integrability of a second order ordinary differential equation known as the generalized Emden-Fowler equation, which was investigated by discrete-group methods. We present 4 classes of general solutions for the parameters obeying the following relations: **A.** σ is arbitrary, $h = 0$; **B.** $\sigma = 1 - h/2$, $0 < h < 2$; **C1.** $\sigma = 1 - h/4$, $0 < h \leq 2$; **C2.** $\sigma = |1 - h|$, $0 < h \leq 2$, $h \neq 1, 4/3$. We discuss the properties of the exact solutions near the initial singularity and at the final stage of evolution.

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1. Introduction

A large class of theories (multidimensional [1, 2, 3], Kaluza-Klein models, supergravity and superstring theories, higher order gravity, see for instance [4],[5]) deal with a weakly coupled scalar field φ with a potential of the form

$$V(\varphi) = \frac{V_0}{\kappa^2} e^{-\sqrt{6}\sigma\kappa\varphi}, \quad (1)$$

where $\kappa = \sqrt{8\pi G_N}$, σ is a dimensionless positive constant, characterizing the steepness of the potential, and the constant V_0 may be positive and negative. A number of authors [4]-[11] (see also references therein) have studied a spationally homogeneous and isotropic Friedmann model containing both a scalar field and a perfect fluid subject to the linear equation of state

$$p = (1 - h)\rho, \quad (2)$$

where the constant h satisfied $0 \leq h \leq 2$. The attention was mainly focussed on the qualitative behaviour of solutions, stability of the exceptional solutions to curvature and shear perturbations and their possible applications within the known cosmological scenaria such as inflation and scaling ("tracking") . In particular, it was found by a phase plane analysis [7, 8, 9] that for "flat" positive potentials ($V_0 > 0$, $0 < \sigma^2 < 1 - h/2$) there exists an unique late-time attractor in the form of the scalar dominated solution. It is stable within homogeneous and isotropic models with non-zero spatial curvature with respect to spatial curvature perturbations for $\sigma^2 < 1/3$ and provides the power-law inflation. For "intermediate" positive potentials ($V_0 > 0$, $1 - h/2 < \sigma^2 < 1$) an unique late-time attractor is the scaling solution, where the scalar field "mimics" the perfect fluid, adopting its equation of state. The energy-density of the scalar field scales with that of the perfect fluid. For $h > 4/3$ this solution is stable within generic Bianchi models to curvature and shear perturbations and provides the power-law inflation. The scaling solution is unstable to curvature perturbations, when $0 < h < 4/3$, although it is stable to shear perturbations. Regions on (σ^2, h) parametrical plane corresponding to various qualitative evolution are presented on figure 1.

Integrability of the model is not well studied yet. Only for the special case for $h = 1$ (dust) and $\sigma = 1/2$ the procedure of getting a solution to the model was given in [11].

In the present paper we study the problem of integrability by quadrature of a spatially flat Friedmann model containing both a minimally coupled scalar field with the exponential potential (1) and a perfect fluid with the equation of state (2) for several classes of sets of (σ, h) -parameters. We apply the methods developed in our previous papers [12]-[14] devoted to integrability of multidimensional cosmological models. It is clear that the possibility of reducing the problem to quadrature depends on the parameters σ and h . We show that in a general case the problem is reduced to integrability of a second order ordinary differential equation known as the generalized Emden-Fowler equation, which was investigated by discrete-group methods [15],[16]. We present here 4 classes of general solutions for the parameters obeying the following relations.

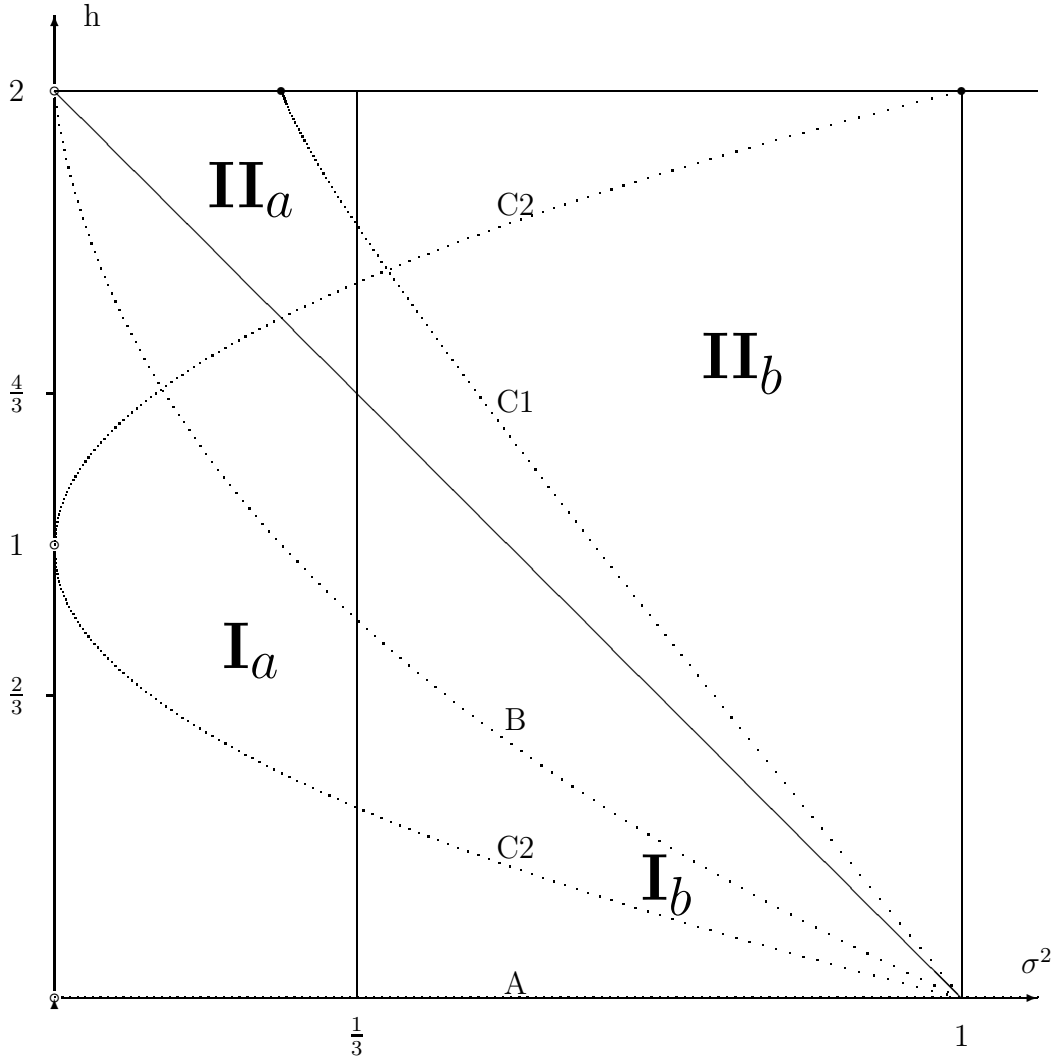


Figure 1. The domains **Ia**, **Ib**, **IIa** and **IIb** are bounded by the lines $h = 0$, $h = 2$, $\sigma^2 = 0$, $\sigma^2 = 1/3$, $\sigma^2 = 1 - h/2$. The solutions corresponding to **Ia** and **IIa** are inflationary on late times with an attractor stable to curvature perturbations. The late-time attractor for **Ia** and **Ib** is the scalar dominated solution and for **IIa** and **IIb** it is the scaling solution. The dotted curves **A**, **B**, **C1** and **C2** are identified in the text and present the integrable by quadrature classes of the model.

- A.** σ is arbitrary, $h = 0$.
- B.** $\sigma = 1 - h/2$, $0 < h < 2$.
- C1.** $\sigma = 1 - h/4$, $0 < h \leq 2$.
- C2.** $\sigma = |1 - h|$, $0 < h \leq 2$, $h \neq 1, 4/3$.

The corresponding curves on (h, σ^2) parametrical plane are presented on figure 1.

The paper is organized as follows. In section 2 we describe the model and obtain the Einstein-scalar field equations in the form of the Lagrange-Euler equations following from some Lagrangian. Dynamical system described by the Lagrangian of this form belongs to the class of pseudo-Euclidean Toda-like systems. To integrate them we apply

in section 3 the methods developed in our papers [12, 13, 14] on multidimensional cosmology. The method used in the general case \mathbf{C} ($\sigma^2 \neq (1 - h/2)^2$, $0 < h \leq 2$) is based on reducing the Euler-Lagrange equations to the generalized Emden-Fowler (second-order ordinary differential) equation. To conclude the paper, we discuss in section 4 the physical properties of the obtained exact solutions. Asymptotic behaviour of these solutions at early and late times is analyzed. Example of the explicit solution in the cosmic time is presented.

2. The model

We start with Einstein Eqs. in a spatially flat Friedmann metric

$$ds^2 = -e^{2\alpha(t)} dt^2 + e^{2x(t)} \left[dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (3)$$

where $\exp[x(t)] \equiv a(t)$ is the scale factor and the function $\alpha(t)$ determines a time gauge ($\alpha(t) \equiv 0$ corresponds to the cosmic time t_c gauge). We assume that the universe contains both a self-interacting scalar field φ with a potential $V(\varphi)$ and a separately conserved perfect fluid with the barotropic equation of state (2). The governing set of equations, which follows from coupled Einstein and scalar field equations, reads

$$\dot{x}^2 = \frac{\kappa^2}{3} \left[\frac{1}{2} \dot{\varphi}^2 + (V(\varphi) + \rho) e^{2\alpha} \right], \quad (4)$$

$$\ddot{x} = \frac{1}{2} (\dot{\alpha} - 3\dot{x}) \dot{x} - \frac{\kappa^2}{2} \left[\frac{1}{2} \dot{\varphi}^2 - (V(\varphi) - p) e^{2\alpha} \right], \quad (5)$$

$$\ddot{\varphi} = (\dot{\alpha} - 3\dot{x}) \dot{\varphi} - V'(\varphi) e^{2\alpha}, \quad (6)$$

$$\dot{\rho} = -3\dot{x}(p + \rho). \quad (7)$$

Using the barotropic equation of state we get immediately from Eq.(7)

$$\rho = \frac{\rho_0}{\kappa^2} e^{-3(2-h)x}, \quad (8)$$

where ρ_0 is an arbitrary positive constant.

It can be easily verified, that the set of Eqs.(4)-(6), where the presence of a perfect fluid density ρ and its pressure p is cancelled by Eqs.(2) and (8), is equivalent to the set of Euler-Lagrange equations obtained from the Lagrangian

$$L(x, y, \alpha, \dot{x}, \dot{y}) = \frac{1}{2} e^{3x-\alpha} (-\dot{x}^2 + \dot{y}^2) - \frac{\kappa^2}{6} e^{\alpha-3x} \left[\rho_0 e^{3hx} + e^{6x} V \left(\frac{\sqrt{6}}{\kappa} y \right) \right], \quad (9)$$

where we introduced the following dimensionless variable

$$y = \frac{\kappa}{\sqrt{6}} \varphi.$$

The equation $\partial L / \partial \alpha = d(\partial L / \partial \dot{\alpha}) / dt = 0$ leads to the constraint Eq.(4). Fixing the gauge $\alpha \equiv F(x, y)$, we can consider Eqs.(5),(6) as the Euler-Lagrange equations obtained from the Lagrangian (9) under the zero-energy constraint (4).

In what follows we consider the potential of the form (1). As the system is symmetric under the transformation $\sigma \rightarrow -\sigma$, $\varphi \rightarrow -\varphi$, without a loss of generality we

will consider only the case $\sigma > 0$. For the exponential potential (1) the Lagrangian (9) looks as follows

$$L(x, y, \alpha, \dot{x}, \dot{y}) = \frac{1}{2}e^{3x-\alpha}(-\dot{x}^2 + \dot{y}^2) - \frac{\kappa^2}{6}e^{\alpha-3x}[\rho_0 e^{3hx} + V_0 e^{6(x-\sigma y)}]. \quad (10)$$

Dynamical system described by the Lagrangian of this form belongs to the class of pseudo-Euclidean Toda-like systems investigated in our previous papers [12, 13, 14]. Methods for integrating of pseudo-Euclidean Toda-like systems are based on the Minkowski-like geometry for characteristic vectors $(\alpha_0 + 3(h-1), 0) \in \mathbb{R}^2$ and $(\alpha_0 + 3, -6\sigma) \in \mathbb{R}^2$ appearing when one puts the gauge $\alpha = \alpha_0 x$ with $\alpha_0 = \text{const}$. Here we do not describe the methods and refer to the above mentioned papers.

3. General solutions

A. σ is arbitrary, $h = 0$.

Here we suppose that the perfect fluid pressure p is equal to its density ρ (Zeldovich-type, or stiff matter). In this case the system with Lagrangian (10) is integrable for arbitrary parameter σ in the so-called harmonic time gauge defined by

$$\alpha(x) = 3x. \quad (11)$$

Let us consider two different cases: $\sigma \neq 1$ and $\sigma = 1$.

If $\sigma \neq 1$ we introduce the following variables

$$u = -\sigma x + y, \quad v = x - \sigma y.$$

In the terms of u and v the equations of motion and the zero-energy constraint look as follows

$$\begin{aligned} \ddot{u} &= 0, \\ \ddot{v} &= V_0(1 - \sigma^2)e^{6v}, \\ -\dot{u}^2 + \dot{v}^2 &= \frac{1 - \sigma^2}{3}[\rho_0 + V_0 e^{6v}]. \end{aligned}$$

We notice that for v we get the Liouville equation. Integrating this set of equations and inverting the linear transformation, we get the following general solution:

the scale factor

$$a \equiv e^x = a_0 \left[f(t - t_0) e^{\sigma A(t-t_0)} \right]^{1/[3(\sigma^2-1)]}, \quad (12)$$

the scalar field

$$\varphi = \frac{\sqrt{6}}{\kappa} \left\{ \ln \left[f^\sigma(t - t_0) e^{A(t-t_0)} \right]^{1/[3(\sigma^2-1)]} + y_0 \right\}, \quad (13)$$

where we introduced the function

$$\begin{aligned} f(t - t_0) &= \sinh(\sqrt{B}|t - t_0|)/\sqrt{B}, \quad V_0(1 - \sigma^2) > 0, \quad B > 0, \\ &= \cosh(\sqrt{B}|t - t_0|)/\sqrt{B}, \quad V_0(1 - \sigma^2) < 0, \quad B > 0, \\ &= \sin(\sqrt{B}|t - t_0|)/\sqrt{B}, \quad V_0(1 - \sigma^2) > 0, \quad B < 0, \\ &= |t - t_0|, \quad V_0(1 - \sigma^2) > 0, \quad B = 0. \end{aligned}$$

The constant B is defined by

$$B = A^2 + 3(1 - \sigma^2)\rho_0.$$

The general solution contains 2 arbitrary constants t_0, A and 2 constants a_0, y_0 , obeying the constraint:

$$e^{6\sigma y_0} = 3a_0^6 |V_0(1 - \sigma^2)|.$$

For the second case $\sigma = 1$ the equations of motion and the zero-energy constraint with respect to the harmonic time gauge read in the old variables

$$\begin{aligned}\ddot{x} &= V_0 e^{6(x-y)}, \\ \ddot{y} &= V_0 e^{6(x-y)}, \\ -\dot{x}^2 + \dot{y}^2 &= -\frac{1}{3} [\rho_0 + V_0 e^{6(x-y)}].\end{aligned}$$

We immediately find the integral of motion $\dot{x} - \dot{y} = \text{const} \equiv 2A$.

If $V_0 > 0$ the constant A is nonzero due to the zero-energy constraint. The general solution in this case for arbitrary V_0 looks as follows

$$a \equiv e^x = \exp \left\{ \left(A + \frac{\rho_0}{12A} \right) (t - t_0) + V_0 [e^{12A(t-t_0)} - 1] + x_0 \right\}, \quad (14)$$

$$\varphi = \frac{\sqrt{6}}{\kappa} \left\{ \left(-A + \frac{\rho_0}{12A} \right) (t - t_0) + V_0 [e^{12A(t-t_0)} - 1] + y_0 \right\}, \quad (15)$$

where the constants A, t_0 are arbitrary and the constants x_0, y_0 obey the relation $y_0 - x_0 = 2At_0$.

For $V_0 < 0$ the additional solution appears (corresponding to $A = 0, \dot{x} = \dot{y}$)

$$\begin{aligned}a \equiv e^x &= \exp \left\{ B(t - t_0) - \frac{\rho_0}{2}(t - t_0)^2 + x_0 \right\}, \\ \varphi &= \frac{\kappa}{\sqrt{6}} \left[\ln a + \frac{1}{6} \ln(-\rho_0/V_0) \right],\end{aligned}$$

where B and x_0 are arbitrary constants. We remind that t is the harmonic time. It is connected with the cosmic time t_c by the differential equation $dt_c = a^3 dt$.

B. $\sigma = 1 - h/2, 0 < h < 2$.

We notice that the model for $\sigma = 1/2$ and $h = 1$ has been integrated in [11]. Here we study a more general case. We fix the time gauge choosing the following function $\alpha(x)$:

$$\alpha(x) = 3(1 - h)x.$$

Then, with respect to the new variables u and v defined by

$$u = \exp \left[\frac{3h}{2}(x - y) \right], \quad v = \exp \left[\frac{3h}{2}(x + y) \right]$$

the equations of motion and the zero-energy constraint may be written in a rather simple form

$$\begin{aligned}\ddot{u} &= 0, \\ \ddot{v} &= \frac{3}{2}h(2 - h)V_0 u^{(4-3h)/h}, \\ \dot{u}\dot{v} &= \frac{3}{4}h^2 [\rho_0 + V_0 u^{2(2-h)/h}].\end{aligned}$$

The set of equations is easily integrable. We obtain

$$u = A(t - t_0) > 0,$$

$$v = \frac{3h^2}{4A^2} \left\{ \rho_0 A(t - t_0) + \frac{h}{4-h} V_0 [A(t - t_0)]^{(4-h)/h} + B \right\} > 0,$$

where A is an arbitrary nonzero constant and B is an arbitrary nonnegative constant, V_0 has arbitrary sign. If $V_0 < 0$, then, the following additional special solution arises

$$u = (-\rho_0/V_0)^{h/[2(2-h)]},$$

$$v = \frac{3}{4} h(2-h) \rho_0^{h/(2-h)} [T^2 - (t - t_0)^2],$$

where the integration constant $T \neq 0$. Then, one easily gets the scale factor

$$a \equiv e^x = (uv)^{1/(3h)} \quad (16)$$

and the scalar field

$$\varphi = \frac{\sqrt{2/3}}{\kappa h} \ln \frac{v}{u}. \quad (17)$$

C. $\sigma^2 \neq (1 - h/2)^2$, $0 < h \leq 2$.

We introduce the following variables for the factorization of the potential in the Lagrangian ([14])

$$u = 3[\sigma x - (1 - h/2)y],$$

$$v = 3[(h/2 - 1)x + \sigma y] - \ln \sqrt{|V_0|/\rho_0}.$$

Then, the equations of motion and the zero-energy constraint for u and v in the harmonic time gauge defined by Eq.(11) look as follows

$$\ddot{u} = \frac{3}{2} h \sigma A_0 (e^{2v} + \varepsilon) \exp \left\{ \frac{h \sigma u + (2 - h - 2\sigma^2)v}{\sigma^2 - (1 - h/2)^2} \right\}, \quad (18)$$

$$\ddot{v} = -\frac{3}{2} A_0 \left(h(1 - \frac{h}{2}) e^{2v} + (2 - h - 2\sigma^2) \varepsilon \right) \exp \left\{ \frac{h \sigma u + (2 - h - 2\sigma^2)v}{\sigma^2 - (1 - h/2)^2} \right\}, \quad (19)$$

$$-\dot{u}^2 + \dot{v}^2 + 3A_0[\sigma^2 - (1 - h/2)^2] (e^{2v} + \varepsilon) \exp \left\{ \frac{h \sigma u + (2 - h - 2\sigma^2)v}{\sigma^2 - (1 - h/2)^2} \right\} = 0, \quad (20)$$

where we denoted

$$A_0 = \rho_0^{-\frac{2-h-2\sigma^2}{2[\sigma^2 - (1-h/2)^2]}} |V_0|^{\frac{h(1-h/2)}{2[\sigma^2 - (1-h/2)^2]}},$$

$$\varepsilon = \text{sgn}(V_0).$$

Let us express \dot{v}^2 from the zero-energy condition (20) as follows

$$\dot{v}^2 = \left[\left(\frac{du}{dv} \right)^2 - 1 \right]^{-1} 3A_0[\sigma^2 - (1 - h/2)^2] (e^{2v} + \varepsilon) \exp \left\{ \frac{h \sigma u + (2 - h - 2\sigma^2)v}{\sigma^2 - (1 - h/2)^2} \right\}. \quad (21)$$

Substituting \ddot{u} , \ddot{v} and \dot{v}^2 into the relation

$$\frac{d^2 u}{d^2 v} = \frac{\ddot{u} - \ddot{v} du/dv}{\dot{v}^2}, \quad (22)$$

we obtain the following second order ordinary differential equation

$$\frac{d^2 u}{dv^2} = \left[\left(\frac{du}{dv} \right)^2 - 1 \right] \left\{ \frac{1}{2} \left(-\frac{\sigma^2 - (1 - h^2/4)}{\sigma^2 - (1 - h/2)^2} + \frac{e^{2v} - \varepsilon}{e^{2v} + \varepsilon} \right) \frac{du}{dv} + \frac{h\sigma/2}{\sigma^2 - (1 - h/2)^2} \right\}. \quad (23)$$

The procedure is valid if $\dot{v} \neq 0$.

The exceptional solution with $\dot{v} \equiv 0$ appears only for the positive potential when $\sigma^2 > 1 - h/2$ and $0 < h < 2$. In the terms of the cosmic time t_c it reads

$$a = \left\{ \sqrt{\frac{3\rho_0}{4(\sigma^2 - (1 - h/2)^2)}} \sigma(2 - h) |t_c - t_c^0| \right\}^{2/[3(2-h)]}, \quad (24)$$

$$\varphi = \frac{\sqrt{2/3}}{\kappa\sigma} \ln \left(\sqrt{\frac{3}{h}} (2 - h) V_0 \sigma |t_c - t_c^0| \right). \quad (25)$$

It should be mentioned, that the set of the equations (18)-(20) does not admit static solutions $\dot{u} = \dot{v} \equiv 0$ as well as the solutions with $\dot{u} = \pm \dot{v}$. So, using the relations (21) and (22) we do not lose any solutions except, possibly, the exceptional solution (24),(25).

Let us suppose that one is able to obtain the general solution to the equation (23) in the parametrical form $v = v(\tau)$, $u = u(\tau)$, where τ is a parameter. Then, we obtain the scale factor $a \equiv \exp[x]$ and the scalar field $\varphi = (\sqrt{6}/\kappa)y$ as functions of the parameter τ . The relation between the parameter τ and the harmonic time t may be always derived by integration of the zero-energy constraint written in the form of the following separable equation

$$dt^2 = \frac{\left(\frac{du}{d\tau} \right)^2 - \left(\frac{dv}{d\tau} \right)^2}{3A_0[\sigma^2 - (1 - h/2)^2] (e^{2v} + \varepsilon)} \exp \left\{ -\frac{h\sigma u + (2 - h - 2\sigma^2)v}{\sigma^2 - (1 - h/2)^2} \right\} d\tau^2. \quad (26)$$

Thus, the problem of the integrability by quadrature of the model is reduced to the integrability of the equation (23). For du/dv it represents the first-order nonlinear ordinary differential equation. Its right hand side is the third-order polynomial (with coefficients depending on v) with respect to the du/dv . An equation of such type is called Abel's equation (see, for instance, [15],[16]).

First of all let us notice that the equation (23) has the partial solution $u = \pm v + \text{const}$ that make the relation (21) singular. However, as was already mentioned, the set of Eqs.(18)-(20) does not admit the solutions with $\dot{u} = \pm \dot{v}$. Existence of this partial solution to the Abel equation (23) allows one to find the following nontrivial transformation

$$e^{2v} = -\varepsilon \frac{X}{Y} \frac{dY}{dX}, \quad (27)$$

$$u = \delta \left[v + \ln \left| \frac{Y}{X} \right| + \ln C \right], \quad \delta = \pm 1, \quad C > 0, \quad (28)$$

that reduces it to the so-called generalized Emden-Fowler equation

$$\frac{d^2 Y}{dX^2} = \text{sgn}[\sigma^2 - (1 - h/2)^2] \left(-\varepsilon \frac{dY}{dX} \right)^l Y^m X^n, \quad (29)$$

where the constant parameters l, m and n read

$$l = \frac{2(\delta\sigma - 1 + h/4)}{\delta\sigma - 1 + h/2}, \quad m = -\frac{\delta\sigma + 1 - h}{\delta\sigma + 1 - h/2}, \quad n = -m - 3. \quad (30)$$

In the special case $l = 0$ Eq.(29) is known as the Emden-Fowler equation.

There are no methods for integrating of the generalized Emden-Fowler equation with arbitrary independent parameters l, m and n . However, the discrete-group methods developed in [16] allow to integrate by quadrature 3 two-parametrical classes, 11 one-parametrical classes and about 90 separated points in the parametrical space (l, m, n) of the generalized Emden-Fowler equation. Further we consider the following integrable classes.

C1. $\sigma = 1 - h/4, 0 < h \leq 2, \delta = 1$.

The parameters l and m given by Eq. (30) are the following

$$l = 0, \quad m = -1 + \frac{2h}{8 - 3h} \in (-1, 1]. \quad (31)$$

The general solution to the generalized Emden-Fowler equation (29) with these parameters reads

$$Y = \frac{\tau}{F_m(\tau)} > 0, \quad X = \frac{1}{F_m(\tau)} > 0,$$

where we introduced the following function

$$F_m(\tau) = \pm \int \left[\frac{2}{m+1} \tau^{m+1} + C_1 \right]^{-1/2} d\tau + C_2. \quad (32)$$

The variable τ changes on the interval which follows from $G_m(\tau) > 0$, where we used the function

$$G_m(\tau) = \varepsilon \left[\frac{F_m(\tau)}{\tau F'_m(\tau)} - 1 \right] \quad (33)$$

equal to the right hand side of Eqs.(27) with substitutions of X, Y and (32). Finally, using Eqs.(27),(28) we find the scale factor

$$a = a_0 \tau^{\frac{4(4-h)}{3h(8-3h)}} G_m^{\frac{2}{3h}}(\tau) \quad (34)$$

and the scalar field

$$\varphi = \frac{\sqrt{6}}{\kappa} \left\{ \ln \left[\tau^{\frac{8(2-h)}{3h(8-3h)}} G_m^{\frac{2}{3h}}(\tau) \right] + y_0 \right\}, \quad (35)$$

where

$$a_0 = C^{\frac{4(4-h)}{3h(8-3h)}} (|V_0|/\rho_0)^{\frac{4(2-h)}{3h(8-3h)}}, \quad y_0 = \ln \left[C^{\frac{8(2-h)}{3h(8-3h)}} (|V_0|/\rho_0)^{\frac{2(4-h)}{3h(8-3h)}} \right].$$

The relation between the variable τ and the cosmic time t_c is the following

$$dt_c^2 = I_0 \tau^{\frac{4(2-h)(4-3h)}{h(8-3h)}} G_m^{\frac{2(2-h)}{h}}(\tau) [F'_m(\tau)]^2 d\tau^2, \quad (36)$$

where

$$I_0 = 16\rho_0^{-\frac{(4-h)^2}{h(8-3h)}} |V_0|^{\frac{4(2-h)^2}{h(8-3h)}} C^{\frac{4(2-h)(4-h)}{h(8-3h)}} / [3h(8-3h)].$$

The general solution (34),(35) (with l and m given by (31)) contains 3 arbitrary constants C , C_1 and C_2 as required.

C2. $\sigma = |1 - h|$, $0 < h \leq 2$, $h \neq 1, 4/3$, $\delta = \text{sgn}(1 - h)$.

Now we have the following parameters in the generalized Emden-Fowler equation

$$l = 3, \quad m = -1 + \frac{h}{4 - 3h}. \quad (37)$$

Its general solution looks as follows

$$X = \frac{\tau}{R_m(\tau)} > 0, \quad Y = \frac{1}{R_m(\tau)} > 0,$$

where the function $R_m(\tau)$ is defined by

$$R_m(\tau) = \pm \int \left[\frac{-2\varepsilon}{|m+2|} \tau^{-m-2} + C_1 \right]^{-1/2} d\tau + C_2, \quad m \neq -2, \quad (38)$$

$$= \pm \int [\varepsilon \ln \tau^2 + C_1]^{-1/2} d\tau + C_2, \quad m = -2. \quad (39)$$

The variable τ changes on an interval, where

$$S_m(\tau) = \varepsilon \left[\frac{R_m(\tau)}{\tau R'_m(\tau)} - 1 \right] > 0. \quad (40)$$

Finally, we get

$$a = a_0 \tau^{\frac{4(1-h)}{3h(4-3h)}} S_m^{\frac{1}{3h}}(\tau) \quad (41)$$

$$\varphi = \frac{\sqrt{6}}{\kappa} \text{sgn}(1 - h) \left\{ \ln \left[\tau^{\frac{2(2-h)}{3h(4-3h)}} S_m^{\frac{1}{3h}}(\tau) \right] + y_0 \right\}, \quad (42)$$

where

$$a_0 = C^{\frac{4(1-h)}{3h(3h-4)}} (|V_0|/\rho_0)^{\frac{2(2-h)}{3h(3h-4)}}, \quad y_0 = \ln \left[C^{\frac{2(2-h)}{3h(3h-4)}} (|V_0|/\rho_0)^{\frac{4(1-h)}{3h(3h-4)}} \right].$$

The variable τ and the cosmic time t_c are related by

$$dt_c^2 = U_0 \tau^{\frac{4(2-3h)}{3h}} S_m^{\frac{2(1-h)}{h}}(\tau) [R'_m(\tau)]^2 d\tau^2, \quad (43)$$

where

$$U_0 = 4\rho_0^{-\frac{4(1-h)^2}{h(3h-4)}} |V_0|^{\frac{(2-h)^2}{h(3h-4)}} C^{\frac{4(2-h)(1-h)}{h(3h-4)}} / (3h|3h-4|).$$

So, this general solution is given in the parametrical form by (41),(42) with l and m from (37). The transition to the cosmic time may be done by solving (43).

4. Properties of solutions

Here we study properties of the obtained exact solutions for positive potentials, though our solutions are valid for any sign of the potential. As the system is symmetrical under the time reflection $t \rightarrow -t$, without loss of generality we only consider expanding near the singularity cosmologies with the Hubble parameter $H = \dot{x} \exp(-\alpha) > 0$.

We introduce the following notation for the relative energy densities

$$\Omega_\rho = \frac{\kappa^2 \rho}{3H^2}, \quad \Omega_{\varphi K} = \frac{\kappa^2 (d\varphi/dt_c)^2}{6H^2}, \quad \Omega_{\varphi P} = \frac{\kappa^2 V(\varphi)}{3H^2}.$$

Due to the constraint (4) the relative energy densities obey the relation

$$\Omega_\rho + \Omega_{\varphi K} + \Omega_{\varphi P} = 1.$$

Also we introduce the scalar field barotropic parameter

$$\frac{p_\varphi}{\rho_\varphi} = \frac{\frac{1}{2} \left(\frac{d\varphi}{dt_c} \right)^2 + V(\varphi)}{\frac{1}{2} \left(\frac{d\varphi}{dt_c} \right)^2 - V(\varphi)}.$$

A1. $0 < \sigma < 1$, $h = 0$.

The general solution is given by Eqs.(12),(13) for $V_0(1 - \sigma^2) > 0$, $B \geq 0$. Near the initial singularity ($t_c \rightarrow +0$) the scale factor is in the main order $a \propto t_c^{1/3}$. There exists the special solution for $A = -\sigma\sqrt{B}$ with $\Omega_\rho \rightarrow 1$ as $t_c \rightarrow +0$. It corresponds to the fluid-dominated solution mentioned in [7, 9, 10]. It gives $p_\varphi/\rho_\varphi \rightarrow 1$ (stiff matter) as $t_c \rightarrow +0$.

For all remaining solutions we obtain $\Omega_\rho \rightarrow 1 - \Omega_{\varphi K}^0$, $\Omega_{\varphi K} \rightarrow \Omega_{\varphi K}^0$, $\Omega_{\varphi P} \rightarrow 0$ as $t_c \rightarrow +0$, where $\Omega_{\varphi K}^0 = (\sigma\sqrt{B} - A)^2/(\sqrt{B} - \sigma A)^2$. Then the barotropic parameter p_φ/ρ_φ tends to -1 , i.e the scalar field is vacuum-like (de Sitter) near the initial singularity.

There is an unique late-time attractor in the form of the scalar field dominated solution. It corresponds to $B = 0$, and $\rho_0 = 0$ in formulas (12),(13) presenting the general solution. The attractor may be written down for the cosmic time t_c

$$a = \tilde{a}_0 t_c^{1/(3\sigma^2)},$$

$$\varphi = \frac{\sqrt{2/3}}{\kappa\sigma} \left[\ln t_c + \ln \sqrt{\frac{3V_0\sigma^4}{1 - \sigma^2}} \right].$$

It is easy to see that for this solution

$$\Omega_\rho = 0, \quad \Omega_{\varphi K} = \sigma^2, \quad \Omega_{\varphi P} = 1 - \sigma^2, \quad \frac{p_\varphi}{\rho_\varphi} = 2\sigma^2 - 1.$$

This attracting solution according to [10] may be called kinetic-potential scaling. For $\sigma^2 < 1/3$ all solutions provide the power law inflation at late times.

A2. $1 < \sigma$, $h = 0$.

The general solution is given by (12),(13) for $V_0(1 - \sigma^2) < 0$, $B > 0$. Behavior near the initial singularity is the same for both general and special solutions with the only difference that the constant $\Omega_{\varphi K}^0$ is the following $\Omega_{\varphi K}^0 = (\sigma\sqrt{B} + A)^2/(\sqrt{B} + \sigma A)^2$.

At the final stage of evolution as $t_c \rightarrow +\infty$ we obtain $\Omega_\rho \rightarrow 1 - \Omega_{\varphi K}^f$, $\Omega_{\varphi K} \rightarrow \Omega_{\varphi K}^f$, $\Omega_{\varphi P} \rightarrow 0$, $p_\varphi/\rho_\varphi \rightarrow 1$, $a \propto t_c^{1/3}$, where $\Omega_{\varphi K}^f = (\sigma\sqrt{B} - A)^2/(\sqrt{B} - \sigma A)^2$. Such behaviour of the scalar field, when it adopts the usual perfect fluid equation of state and its energy-density scales with that of the perfect fluid, is called scaling (or sometimes "tracking").

B and **C2** in regions **Ia** and **Ib**.

Behaviour near the initial singularity was described in [7, 9, 10] using qualitative methods. There exists the fluid dominated solution with $a \propto t_c^{2/[3(2-h)]}$ and $\Omega_\rho \rightarrow 1$

as $t_c \rightarrow +0$. All remaining solutions describe the domination of the kinetic contribution of the scalar field: $\Omega_\rho \rightarrow 0$, $\Omega_{\varphi K} \rightarrow 1$, $\Omega_{\varphi P} \rightarrow 0$, $p_\varphi/\rho_\varphi \rightarrow 1$, $a \propto t_c^{1/3}$.

The behaviour at late times is the same as one for **A1**.

C1 and **C2** in regions **IIa** and **IIb**.

Behaviour near the initial singularity is the same as in the previous case. The late-time attractor for $h \in (0, 1)$ is the special solution described by Eqs.(24) and (25). For these solution we have

$$\Omega_\rho = 1 - \frac{2-h}{2\sigma^2}, \quad \Omega_{\varphi K} = \frac{(2-h)^2}{4\sigma^2}, \quad \Omega_{\varphi P} = \frac{h(2-h)}{4\sigma^2}, \quad \frac{p_\varphi}{\rho_\varphi} = 1 - h.$$

This is a typical scaling behaviour.

One of the examples of this behaviour may be given explicitly: the exact solution of the class **C1** with $h = 2$ and $\sigma = 1/2$ for the positive potential reads with respect to the cosmic time

$$a = a_0 e^{\sqrt{\Lambda/3}t_c} \left\{ \frac{1 - Ae^{-\sqrt{3\Lambda}(t_c-t_c^0)}}{1 + Ae^{-\sqrt{3\Lambda}(t_c-t_c^0)}} \left[\frac{\sqrt{3\Lambda}}{2}(t_c - t_c^0) - \frac{2A(1 - Ae^{-\sqrt{3\Lambda}(t_c-t_c^0)})}{(1+A)(1 + Ae^{-\sqrt{3\Lambda}(t_c-t_c^0)})} \right] \right\}^{1/3},$$

$$\varphi = \frac{\sqrt{6}}{\kappa} \ln \left\{ \frac{V_0}{\Lambda} \left[\frac{1 + Ae^{-\sqrt{3\Lambda}(t_c-t_c^0)}}{1 - Ae^{-\sqrt{3\Lambda}(t_c-t_c^0)}} \left(\frac{\sqrt{3\Lambda}}{2}(t_c - t_c^0) - \frac{1-A}{1+A} \right) - 1 \right] \right\}^{1/3},$$

where $\Lambda \equiv \rho_0$ is the cosmological constant. The solution contains 3 integration constants: arbitrary t_c^0 , positive a_0 and A obeying $|A| < 1$. The late-time attractor corresponds to $A = 0$. For this attracting solution we get: $\Omega_\rho \rightarrow 1$, $\Omega_{\varphi K} \rightarrow 0$, $\Omega_{\varphi P} \rightarrow 0$, $p_\varphi/\rho_\varphi \rightarrow -1$, $a \propto t_c^{1/3} \exp \sqrt{\Lambda/3}t_c$, $H \rightarrow \sqrt{\Lambda/3}$ as $t_c \rightarrow +\infty$.

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